Document ID: 04_01_13_1
Date Received: 2013-04-01 **Date Revised**: 2013-07-24 **Date Accepted**: 2013-08-16
Curriculum Topic Benchmarks: M4.4.8, M5.4.12, M8.4.2, M8.4.15, S15.4.4, S.16.4.1, S16.4.3
Grade Level: High School [9-12]
Subject Keywords: circumference, radius, Eratosthenes, gnomon, sun dial, Earth, noon, time zone, latitude, longitude, collaboration
Rating: Advanced

How Big Is the Earth?

By: Stephen J Edberg, Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, M/S 183-301, Pasadena CA 91011
e-mail: Stephen.J.Edberg@jpl.nasa.gov

From: The PUMAS Collection http://pumas.jpl.nasa.gov

Eratosthenes, the third librarian of the Great Library of Alexandria, measured the circumference of Earth around 240 B.C.E. Having learned that the Sun passed through the zenith[1] on the summer solstice[2] as seen in modern day Aswan, he measured the length of a shadow on the solstice in Alexandria. By converting the measurement to an angle he determined the difference in latitude – what fraction of a circle spanned the separation – between Aswan and Alexandria. Knowing the physical distance between Aswan and Alexandria allowed him to determine the circumference of Earth.

Cooperating schools can duplicate Eratosthenes' measurements without the use of present day technology, if desired. Sharing their data permits students to calculate the circumference and the radius of Earth. The measurements do not require a site on the Tropic of Cancer (or the Tropic of Capricorn) but they must be made at local solar noon on the same date. The schools' separation must be known and the other school should be as close as possible to due north or south of your campus. (If the schools are at significantly different elevations, this will produce some error; having a larger distance between the schools will reduce the error.)

---

[1] The zenith is the point in the sky directly overhead. Most people assume it is lower in the sky than it actually is. Looking up, slowly spin your body in place: the axis of rotation of the sky marks the zenith.
[2] The summer solstice marks the maximum points above and below the equator in the Sun's annual journey through the sky. Summer is defined to begin in the northern and southern hemispheres, respectively, when it reaches those maximum points. The latitudes of these maxima, approximately 23.5° North and South, define the parallels of latitude known as the Tropic of Cancer in the north and the Tropic of Capricorn in the south.

OBJECTIVE: Make measurements of the lengths of shadows on the same date at local noon[3]. From these measurements, determine the circumference and radius of Earth.

APPARATUS: (1) Build and install, or find on campus, a **gnomon** (pronounced noh-mun) to cast the shadow whose length will be measured. Your gnomon may be as simple as a dowel, flagpole, volleyball net support, or fence post. It is important that it be truly vertical. A circular cross-section is desirable so the end of the shadow is easily interpreted. Make sure the gnomon is positioned on a flat, truly horizontal surface large enough for the gnomon's noontime shadow to fall on it during all seasons (the shadow will be longest at the winter solstice). There should be an unobstructed view of the Sun at local noon (which can be different by more than an hour from clock [Standard Time] noon). Higher gnomons are more impressive but also have "fuzzier," less distinct shadows at their ends, making the length measurement more ambiguous. Use a **carpenter's level** to make measurements of 90° at two positions separated by about 90° around the gnomon to verify the gnomon is truly along a radius from Earth's center (i.e., vertical, perpendicular to the ground). The carpenter's level can also confirm that the surface around the gnomon is flat and level (thus the gnomon perpendicular [normal] to the surface).

(2) A **tape measure** will be used to measure the height of the gnomon and the length of its shadow on the same plane as its base.

(3) A **scientific calculator** (or spreadsheet) for each student or group of students.

(4) If desired: A **sundial**[4] will provide local solar noon for any date during the day(light). The gnomon itself serves as a sun dial if time marks are made on the ground.

(5) Alternate approach to calculate the clock time of noon: **Map or a Global Positioning System (GPS) unit** (or cellular telephone with the appropriate app) is used to determine your school's latitude and longitude and azimuth to the collaborating school. See THE UNDERLYING PRINCIPLES below for the calculation.

PROCEDURE:

For a measurement of Earth's circumference, same-day local solar noon measurements of the length of the gnomon's shadow must be made at two locations with a known difference in latitude. A greater latitude difference and a lesser longitude difference will improve the accuracy of the final result. The following steps need to be followed at both collaborating schools.

---

[3] Local noon is measured with a sundial, not a clock face. Given a school's longitude, the clock time of local noon is easily computed or it can be determined by observation.

[4] A sundial is a calibrated device whose gnomon casts a shadow on numerals indicating time of day. Often the gnomon is tilted to the latitude of the sundial's position on Earth. If the sundial's design includes a tilted gnomon, the sundial must be aligned so the gnomon parallels the local north-south meridian. In other words, the gnomon points north. More generally, the tilted gnomon of a sundial should point toward the celestial pole of the hemisphere, north or south, in which it is sited.

Establish the gnomon, in place and vertical. Measure the height of the gnomon.

Determine the clock time of local noon if a sundial is not being used. This can be accomplished mathematically with GPS or map coordinates and the formulas in THE UNDERLYING PRINCIPLES or by observation.

To do this by observation, in advance of the coordinated observation date calibrate the gnomon by observing the clock time when the gnomon's shadow is shortest. Observe the gnomon's shadow for a span of about two hours, marking the ground with chalk or tape every 10 minutes, to confirm the clock time when its shadow is shortest. When daylight saving time is in use, the time of local noon can be over an hour later than clock-noon. (Note: The standard clock time of local noon changes significantly over monthly intervals due to the shape of Earth's orbit around the Sun. Measure your local noon on a date as close as possible to the day of coordinated observation.) Alternatively, the clock time of local noon can be calculated, as mentioned above, and then confirmed during the observations of the gnomon's shadow length.

Confirm the date when shadow measurements will be made with the partnering school, including alternate dates if weather or other events interfere on the primary date. The partner school will need to have made the same preparations as described for your school.

At local noon on the selected date, measure the length of the shadow of the gnomon.

Now the calculations begin. The length of the shadow (indicated with s in Figure 1) and the height of the gnomon (h in Fig. 1) form two sides of a right triangle. The angle at the top, ψ (psi), between the hypotenuse and the gnomon, is the one of interest (Fig. 1).

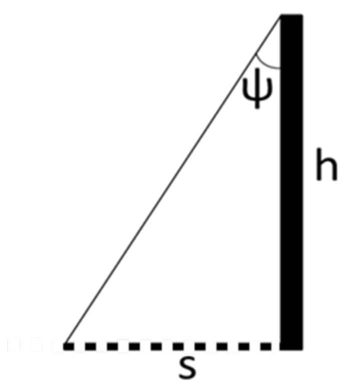

Fig. 1. The tangent of angle ψ is, by definition, the (opposite side)/(adjacent side) = (shadow's length)/(gnomon's height): $\tan \psi = s/h$.

Using the inverse tangent function, calculate ψ.

$$\psi = \arctan (s/h) \qquad (1)$$

(Be sure your calculator is giving the arctan result in degrees rather than radians and uses degrees for inputs in other calculations. Some spreadsheets use radians for inputs/outputs, and each must be converted from/to degrees.)

Determine the absolute value of the difference in the angles ψ_1 & ψ_2 for the two schools,

$$\Delta\psi = |\psi_1 - \psi_2| \quad (2)$$

which is the difference in latitude between the two schools if we assume that the sun beams are parallel to each other (Figure 2).

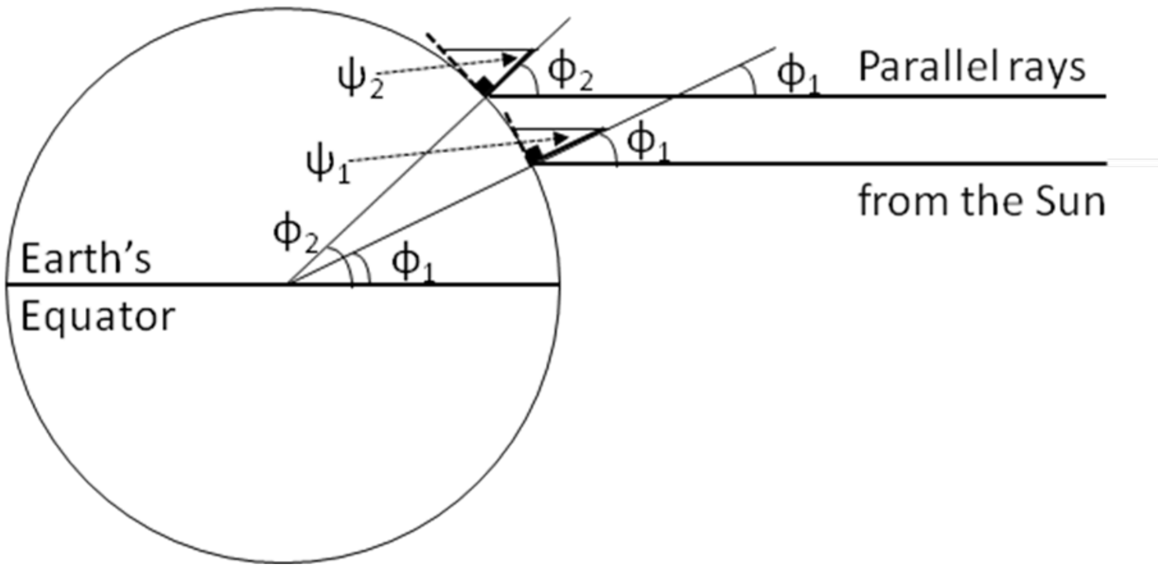

Fig. 2. In this cross-section of Earth, the latitudes of the collaborating schools, ϕ_1 and ϕ_2, are measured relative to Earth's equator from its center. Dashed line segments are tangent to Earth's surface at each latitude, and represent the local horizontal surface where s is measured in Fig. 1. Solid line segments paralleling the Sun's rays represent the edge of a gnomon's shadow from the top of the gnomon to the ground surface. The gnomons are bold segments extending along the radial lines from Earth's center and have height h in Fig. 1. The small triangles that include ψ_1 and ψ_2 are tilted, miniature versions of Fig. 1. Notice that angles ψ_1 and ψ_2 are different at each school site because of their different latitudes. The difference in latitudes,
$\Delta\phi = |\phi_1 - \phi_2| = \Delta\psi = |\psi_1 - \psi_2|$.

The fraction of the circumference (F) of Earth spanned by the latitudes of the two schools is

$$F = \Delta\psi/360 \quad (3)$$

If the schools are directly north/south of each other, the flight distance (D[km]) between them (in kilometers or miles) multiplied by 1/F gives the circumference of Earth (C).

$$C[km] = D[km]/F \quad (4)$$

Finding Earth's radius (R) is easily accomplished by dividing the circumference (C) by 2π:

$$R[km] = C[km]/2\pi \quad (5)$$

If the schools do not fall directly along a north-south meridian, more calculation is necessary. The physical distance between schools, the flight distance D[km], is known, measured from a map, for instance. By calculating the angular distance between the schools (Equations 6 and 7) the SCALE in [km/degree] can be calculated (Equation 8). Then, knowing the schools' latitudes, the difference in their (angular) latitudes can be converted to the physical separation of their latitudes L[km] (Equation 9). Use L in equation (4) in place of D and compute the Earth's circumference, as before.

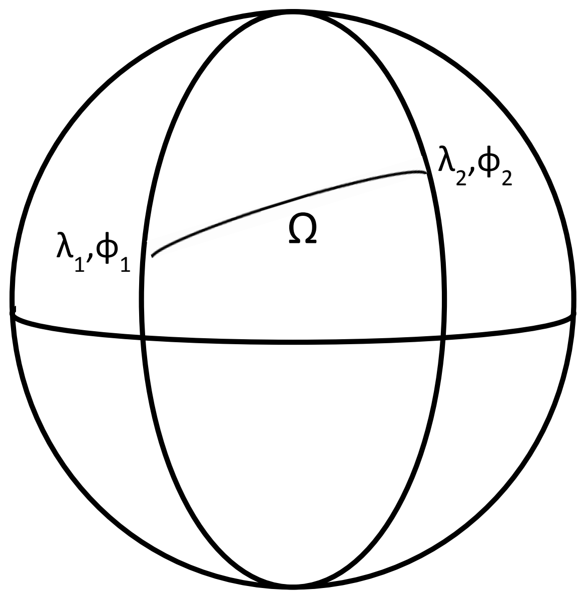

Fig. 3. On this stylized globe, two sites and their separation are indicated. Each site is on a meridian of longitude (a great circle), with its longitude (λ) and latitude (ϕ) indicated. Δ is an arc of a great circle as well and is the minimum angular and physical distance between the two sites. The two arcs ($90 - \lambda_{1,2}$) and Ω are a spherical triangle that can be solved for the angular length of Ω[°].

To determine L, the angular separation of the schools, omega = Ω[°], can be calculated using spherical trigonometry's Law of Cosines:

$$\cos(\Omega[°]) = \sin(\phi_2) \times \sin(\phi_1) + \cos(\phi_2) \times \cos(\phi_1) \times \cos(\lambda_2 - \lambda_1) \quad (6)$$

where:
ϕ = latitude [in decimal degrees, $\pm$dd.mm.ss]
Latitudes in the northern hemisphere are positive; in the southern hemisphere they are negative.

λ = longitude [in decimal degrees, $\pm$dd.mm.ss]
Use the same algebraic sign for the longitudes as long as they don't straddle the Prime Meridian at 0° longitude or the anti-meridian at 180° longitude. If Ω crosses either of these meridians, assign + to one side and - to the other and proceed, taking into account the signs algebraically.

Take the inverse cosine of the solution to (6) to get the angular separation Ω[°].

$$\Omega[°] = \arccos[\sin(\phi_2) \times \sin(\phi_1) + \cos(\phi_2) \times \cos(\phi_1) \times \cos(\lambda_2 - \lambda_1)] \quad (7)$$

The scale determined from the angular and physical separations,

$$\text{SCALE} = D[\text{km}]/\Omega[°] \quad (8)$$

can be applied to the difference in the schools' latitudes. As noted in Figure 2's caption $\Delta\phi = |\phi_1 - \phi_2|$ (and $= \Delta\psi = |\psi_1 - \psi_2|$; the EXTENSION gives hints for a geometric proof), so

$$L[\text{km}] = \Delta\phi \times \text{SCALE} \quad (9)$$

Of course, having the physical distance and the angular separation of sites on a great circle can immediately be turned into Earth's circumference. Eratosthenes did not have the use of spherical triangle calculations (the rudiments of spherical trigonometry were still about two centuries in his future). It's not known how he determined the physical distance between Alexandria and Aswan. Since using a surveyor's chain is very time consuming and not very practical for actual distance measurements between schools, we use more modern mathematics to determine a scale and permit the schools to have somewhat different longitudes (as well as different latitudes) for the shadow measurements and make the observations to demonstrate the validity of Eratosthenes' method.

THE UNDERLYING PRINCIPLES:

Eratosthenes may have been inspired by the fact that on the summer solstice in Aswan, Egypt, a viewer's shadow obscured the reflection of the Sun going down a well at local noon. There on the Tropic of Cancer, a gnomon would exhibit no shadow whereas further north in Alexandria a measurement of the shadow would give the difference in latitude between Aswan and Alexandria immediately because ψ_2 in Eq. 2 is equal to zero.

It would then be easy to calculate Earth's circumference, even without knowing the actual latitudes of the two cities.

In reality, any pair of known latitudes can be used directly with equations (2)-(5) to calculate Earth's circumference and radius.

Determining Local Solar Time without a sun dial:

{Note: If you use the minimum length of the gnomon's shadow to determine local solar noon, as described in the Procedure section above, the time difference between local solar noon and the local clock time is all you really need to calculate Local Solar Time.}

Standard time zones are defined every 15° of longitude, counted from the Prime Meridian at 0° that runs through Greenwich, England[5]. They span 7.5° both east and west of the zone meridian. Determine the difference between your longitude (abbreviated "long;" use GPS or a map)[6] and the nearest time zone longitude (TZlong, it will be a multiple of 15°; see the table below). Then calculate the difference between Standard Time and Local Solar Time, LSoT. Sometimes Daylight Saving Time (called Summer Time in some places) must also be included in the calculation. Confirm that longitudes are converted to decimal degrees. Operations using degrees-minutes-seconds instead of decimal degrees can be used but must be done and reduced correctly.

U.S. & North American Time Zones

Zone Name	Hawaii	Alaska	Pacific	Mountain	Central	Eastern	Atlantic
Central Longitude	165° West	150° West	120° West	105° West	90° West	75° West	60° West
Time Difference From 0°	-10 hours	-9 hours	-8 hours	-7 hours	-6 hours	-5 hours	-4 hours
Daylight Saving Time Difference	Not adopted	-8 hours	-7 hours	-6 hours	-5 hours	-4 hours	-3 hours

$$\Delta \text{Long} = \text{TZlong} - \text{long} \tag{10}$$

The longitude difference turns into a time difference with the conversion 1° = 4 minutes. To demonstrate this, consider that the Sun goes from noon one day to noon the next, making a full 360° circle above and below the horizon over 24 hours:
360°/24 hours can be converted as follows.

[5] Political time zones on maps usually have more irregular boundaries and spans.
[6] Many GPS units can be set up to provide a good value of longitude (and latitude) right on their screens. Smart phone apps are available for reading longitude and latitude from the phone's built in GPS receiver. Alternatively, paper maps can be measured and read. On-line maps, like those at http://www.mytopo.com/ and at http://www.noaa.gov/ can be used; see DATA SOURCES below for instructions.

24 hours x [60 minutes/1 hour] = 1440 minutes
360°/24 hours = 360°/1440 minutes
Dividing both numerator and denominator of 360°/1440 minutes by 360 yields
1°/4 minutes.

$$\Delta t = \Delta Long \text{ x } 4 \text{ [min]} \quad (11)$$

$$ST = \Delta t + LSoT \quad (12)$$

Where:
ΔLong = Difference of the longitude of the standard time zone and the gnomon's longitude[7]
TZlong = time zone longitude
long = gnomon longitude
Δt = Difference from Standard Time
LSoT = Local Solar Time
ST = Standard Time

Example A: You plan to measure the Sun's altitude at noon (12:00) Local Solar Time in February. Your gnomon's longitude is 78°34' west = -78.5667° so your time zone longitude is -75°.

$$\Delta Long = -75° - (-78.5667°) = +3.5667° \quad (10a)$$

$$\Delta t = +3.5667 \text{ x } 4 = +14.2667 \text{ [min]} \quad (11a)$$

$$ST = 14.2667 \text{ [min]} + 12{:}00 = 12{:}14.2667 = 12{:}14{:}16 \text{ local Standard Time} \quad (12a)$$

This is the Standard Time at which a sun dial will show 12:00 noon at this site.

Example B: You plan to measure the Sun's altitude at noon (12:00) Local Solar Time in May. Your gnomon's longitude is 82°36' west = -82.60° so your zone longitude is -90°.

$$\Delta Long = -90° - (-82.60°) = -7.40° \quad (10b)$$

$$\Delta t = -7.40 \text{ x } 4 = -29.60 \text{ [min]} \quad (11b)$$

$$ST = -29.60 \text{ [min]} + 12{:}00 = 11{:}30.40 = 11{:}30{:}24 \text{ local Standard Time} \quad (12b)$$

This is the Standard Time at which a sun dial will show noon at this site. But Daylight Saving Time is in effect: "Spring forward, Fall back." Add one hour to the Standard Time: The sun dial will show noon at 12:30:24 on the clock.

---

[7] It is very easy to get snagged by this calculation. Longitudes WEST of Greenwich are NEGATIVE and subtractions of negative values may occur. The examples illustrate some variations.

DISCUSSION:

Measurements at local noon are required to separate the mixed components of shadow length that would occur at other times of day. Because of the Sun's apparent annual motion around the sky (a reflection of Earth's orbital motion), shadow measurements need to be made the same day. At some times of year there can be measurable changes in shadow length from one noon to the next.

Greater latitudinal separation reduces the effects of measurement errors, be they due to the soft shadow edge or uncertainties in the physical distance between measurement sites. Lesser longitudinal separation reduces the effect on the SCALE.

The SCALE calculated in equation (8) includes an inherent error (leading to an error in Earth's circumference) if the longitudes of the observing sites are different. In general, the SCALE will be within 10% of the true value if the schools meet the following criteria:

1. The school separations are within $D < 1000$ km.
2. The azimuth from the southern school to the northern school (use a map or the spherical trigonometry Law of Cosines) is less than 25° east or west of true north.

EXTENSIONS:

<u>Latitude Determination</u>

The school's latitude can be measured at night using a **protractor, thread or fishing line, and a spare hexagonal nut** (or a few paperclips serving as a plumb bob on the thread). Be sure the thread pivots on the measuring center of the protractor, as illustrated in Figure 4. Make photocopies of the protractor on card stock and distribute the parts to all students so they can assemble the instrument themselves. The thread should pivot from the center of the protractor's semi-circle of angles.

The observer looks along the flat edge of the protractor towards a pole star (Polaris in the northern hemisphere, faint Sigma Octantis in the southern hemisphere). Once the plumb bob has settled, a companion with a flashlight can read the angle or a thumb and finger can trap the thread in place and the angle can be read indoors. Common protractor designs will require the observer to subtract the measured angle from 90° to give the observer's latitude. Combined with another observer's latitude, the calculation follows equations (2)-(5) as above.

The pole stars in both hemispheres are not exactly centered on their respective celestial poles (Polaris is about 1° off, Sigma Octantis is a little closer). This will throw off the results.

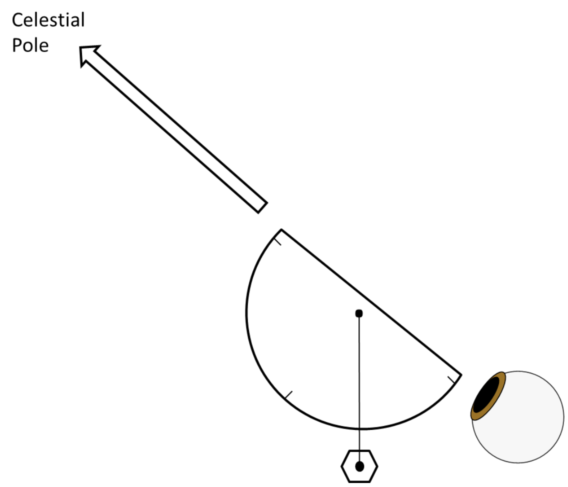

Fig. 4. The addition of a weighted thread to the measuring center of a protractor makes a simple angle measurement device. By looking along the base of the protractor at the celestial pole, the thread indicates the observer's latitude. The thread should pivot from the center of the protractor's semi-circle of angles. (Note that the indicated value may have to be subtracted from 90° to represent latitude, depending on the protractor's design. Remember that making the protractor's flat side horizontal is the equivalent of being at the equator where the latitude is 0° and most protractors will read 90°.)

Latitude Difference

Have students generate a geometric proof that $\Delta\phi = |\phi_1 - \phi_2| = \Delta\psi = |\psi_1 - \psi_2|$ from Figure 2. Hints: Make a sketch. The key is to determine/compare all the angles in each triangle (ψ, 90, 90- ψ) and along a parallel ray (ϕ, 90, 180-[90-ϕ]). Add a perpendicular from the parallel ray to the end of the shadow and compare the values of opposite angles.

DATA SOURCES:

http://www.mytopo.com/ presents topographic maps of the United States. Pick a location (your city's name and state) and search for it. Zoom in until your school is visible. Center and then enlarge it further and move the cursor to the position of your gnomon and read its latitude and longitude just outside the lower left corner of the map.

A similar, slightly clumsier procedure can be followed at http://www.noaa.gov/. Search by city and state and then scroll down on the page to the map on the right side. Demagnify, if necessary, and drag the map area around to find your school site, magnify it, and then click on the gnomon position to specify the "Requested Location." A less precise position with elevation appears above the map, specified as the area of the "Point Forecast," with the "Forecast Area" in a green quadrilateral.

Many people know that the North Star, Polaris, can be found by using the "Pointers" on the Big Dipper. One source of directions, among many, is

http://earthsky.org/tonight/use-big-dipper-to-find-polaris-the-north-star.

Finding Sigma Octantis, is more difficult due to its faintness. Instructions can be found here (and elsewhere on the World Wide Web):

http://assabfn.blogspot.com/2010/08/find-south-celestial-pole-scp.html.

To find either of these pole stars, start looking after twilight is complete. One's eyes should have time to dark adapt before searching. The sky should be cloud-free, transparent, and the horizon in the pole star's direction should be free of obstructions. Observers at lower latitudes may have more difficulty finding these stars.

A very useful discussion, with examples of spherical trigonometry including equation (6), can be found at http://www.krysstal.com/sphertrig.html.

ACKNOWLEDGMENTS:

Suggestions from an anonymous reviewer and the PUMAS editor improved the clarity and completeness of this presentation.

This publication was prepared by the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.