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Finding All Solutions to a Puzzle

By: Ted Bainbridge, Ph.D., retired email: 4thecho@gmail.com

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Some of the power of algebra comes from the ability to explore relationships among variables, instead of just finding one solution to one problem. Studying those relationships gives us a better understanding of the problem that interests us.

Many problems have several solutions. Algebra finds those solutions and reveals their nature whether there are two, many, or an infinity of them. Sometimes algebra proves that there are no solutions at all, thus saving extra and worthless effort.

If we find multiple solutions, we can choose one that fits some other requirement we want to impose on the situation. For example, some engineering problems have multiple solutions and it is logical to choose the one that costs the least to build.

Some people find the following puzzle difficult to solve, even after many trial-and-error efforts have exhausted them. Algebra guarantees that we will find a solution if one exists, proves how many solutions exist, and shows us all of those solutions.

Here is the puzzle: arrange the non-zero digits in a 3-by-3 array so that the numbers in each column, the numbers in each row, and the numbers on each diagonal add up to the same total.

Without solving the puzzle can you determine what each row, column, and diagonal totals? Try it before reading on.

The integers from 1 through 9 total 45. Since each of the three rows has the same total, each row must total 1/3 of 45, or 15.

Here is a solution:

4	3	8
9	5	1
2	7	6

There are other solutions. We can interchange rows 1 and 3. (The content of the rows doesn't change. The content of the columns doesn't change. The content of the diagonals doesn't change.) Alternatively, we can interchange columns 1 and 3 with similar results. Since we can create 2 solutions by interchanging rows and 2 by interchanging columns, we can create a total of 2 x 2 = 4 solutions.

Are there any other solutions? Let's use algebra to find out. We will do the following:

1. Write all the equations that describe the puzzle.
2. Eliminate as many variables as possible from the equations.
3. Notice that solving a small part of the problem immediately leads to a solution of the entire problem.
4. Find all solutions to the small problem. For each such solution, find the solution to the original puzzle.

<u>Step 1 (write equations)</u>

Assign a variable to each location in the array. Then write the equations that describe the original puzzle.

a	b	c
d	e	f
g	h	j

Equation A: $a + b + c = 15$ Equation E: $b + e + h = 15$
Equation B: $d + e + f = 15$ Equation F: $c + f + j = 15$
Equation C: $g + h + j = 15$ Equation G: $a + e + j = 15$
Equation D: $a + d + g = 15$ Equation H: $c + e + g = 15$

Having 8 linear equations in 9 variables makes the system "underdetermined". One variable will be "free" because it won't be determined by the equations. Each value that we choose for that variable will create another solution to the puzzle.

<u>Step 2 (eliminate variables)</u>

Solve Equation A for b, solve Equation C for h, solve Equation D for d, and solve Equation F for f:

Equation A: $a + b + c = 15 \Rightarrow b = 15 - a - c$ (Equation A*)
Equation C: $g + h + j = 15 \Rightarrow h = 15 - g - j$ (Equation C*)
Equation D: $a + d + g = 15 \Rightarrow d = 15 - a - g$ (Equation D*)
Equation F: $c + f + j = 15 \Rightarrow f = 15 - c - j$ (Equation F*)

Substitute A*, C*, D*, and F* into Equations B, E, G, and H:

Equation B: $d + e + f = 15$

$(15 - a - g) + e + (15 - c - j) = 15$

$15 + e = a + g + c + j$ (Equation B*)

Equation E: $b + e + h = 15$

$(15 - a - c) + e + (15 - g - j) = 15$
$15 + e = a + c + g + j$ which is the same as B*

Equation G: $a + e + j = 15 \Rightarrow a + j = 15 - e$ (Equation G*)

Equation H: $c + e + g = 15 \Rightarrow c + g = 15 - e$ (Equation H*)

The original system of 8 equations in 9 unknowns has been reduced to a system of 3 equations (B*, G*, and H*) in 5 unknowns. That tells us 2 variables can have multiple values instead of being specified in a unique solution to the problem.

<u>Step 3 (identify small problem)</u>

Notice that finding acceptable values for a, b, d, and e immediately gives a solution to the original puzzle:

a	b	c
d	e	f
g	h	j

a and b determine c (by Equation A)
d and e determine f (by Equation B)
a and d determine g (by Equation D)
b and e determine h (by Equation E)
c and f determine j (by Equation F)

<u>Step 4 (solve small problem)</u>

<u>Step 4 part 1 (find some useful relationships)</u>

Substituting expressions from G* and H* into B* gives us a new form of Equation B*: $15 + e = a + g + c + j$

$15 + e = (a + j) + (c + g)$
$15 + e = (15 - e) + (15 - e)$
$3e = 15$
$e = 5$

Then Equation H* becomes $c + g = 15 - e = 15 - 5 = 10$

Equations A* and D* tell us $b + d = (15 - a - c) + (15 - a - g)$
$= 30 - 2a - c - g$

Then Equation H* tells us $b + d = 30 - 2a - 10$
$= 20 - 2a$ (Equation H**)

<u>Step 4 part 2 (describe solution process)</u>

We will search for solutions to the small problem by considering all possible values of a, b, d, and e:

a	b	
d	e	

The solution process will be:

1. Set e = 5 as required by Equation B*.
2. Treat a as a free variable, setting it to 1, 2, 3, 4, 6, 7, 8, and 9 in turn.
   (a = 5 is not allowed because e = 5.)
3. For each allowable value of a, let that value determine values of b and d by using Equation H**: b + d = 20 - 2a.
4. For each allowable pair of values b and d, determine if all of the original eight equations are or are not satisfied simultaneously.

Step 4 part 3 (what if a = 1 ?)

If a = 1 then b + d = 20 - 2a = 20 - 2 = 18.
No two distinct digits total 18, so a = 1 is not allowed.

Step 4 part 4 (what if a = 3 ?)

If a = 3 then b + d = 20 - 2a = 20 - 6 = 14.
The only distinct digits that total 14 are 9-and-5 and 6-and-8.
Notice that 9-and-5 is not allowed because e = 5.
If b = 6 then: Equation A ⇒ c = 15 - a - b = 15 - 3 - 6 = 6 = b.
That is not allowed so b ≠ 6.
If b = 8 then: Equation D ⇒ g = 15 - a - d = 15 - 3 - 6 = 6 = d.
That is not allowed so b ≠ 8.
So 6-and-8 is not allowed. So a = 3 is not allowed.

Step 4 part 5 (what if a = 7 ?)

If a = 7 then b + d = 20 - 2a = 20 - 14 = 6.
The only distinct digits that total 6 are 1-and-5 and 2-and-4.
Notice that 1-and-5 is not allowed because e = 5.
If b = 2 & d = 4 then: D ⇒ g = 15 - a - d = 15 - 7 - 4 = 4 = d.
That is not allowed so b ≠ 2.
If b = 4 then: Equation A ⇒ c = 15 - a - b = 15 - 7 - 4 = 4 = b.
That is not allowed so b ≠ 4.
So 2-and-4 is not allowed. So a = 7 is not allowed.

Step 4 part 6 (what if a = 9 ?)

If a = 9 then b + d = 20 - 2a = 20 - 18 = 2.
No distinct digits total 2, so a = 9 is not allowed.

Step 4 part 7 (what if a = 2 ?)

If a = 2 then b + d = 20 - 2a = 20 - 4 = 16.
The only distinct digits that total 16 are 7-and-9.

If b = 7 and d = 9 then we have a correct solution:

2	7	6
9	5	1
4	3	8

If b = 9 and d = 7 then we have another correct solution:

2	9	4
7	5	3
6	1	8

<u>Step 4 part 8 (what if a = 4 ?)</u>

If a = 4 then b + d = 20 - 2a = 20 - 8 = 12.
The only distinct digits that total 12 are 4-&-8, 5-&-7, 3-&-9.
Notice that 4-and-8 cannot be allowed because a = 4.
Notice that 5-and-7 cannot be allowed because e = 5.
If b = 3 and d = 9 then we have another correct solution:

4	3	8
9	5	1
2	7	6

If b = 9 and d = 3 then we have another correct solution:

4	9	2
3	5	7
8	1	6

<u>Step 4 part 9 (what if a = 6 ?)</u>

If a = 6 then b + d = 20 - 2a = 20 - 12 = 8.
The only distinct digits that total 8 are 2-and-6, 3-&-5, 1-&-7.
Notice that 2-and-6 cannot be allowed because a = 6.
Notice that 3-and-5 cannot be allowed because e = 5.
If b = 1 and d = 7 then we have another correct solution:

6	1	8
7	5	3
2	9	4

If b = 7 and d = 1 then we have another correct solution:

6	7	2
1	5	9
8	3	4

Step 4 part 10 (what if a = 8 ?)

If a = 8 then b + d = 20 - 2a = 20 - 16 = 4.
The only distinct digits that total 4 are 1-and-3.
If b = 1 and d = 3 then we have another correct solution:

8	1	6
3	5	7
4	9	2

If b = 3 and d = 1 then we have another correct solution:

8	3	4
1	5	9
6	7	2

That completes our search. We proved that there are eight correct solutions and no others. Inspection shows that each solution is unique. Arranging the solutions as follows illuminates relationships among them.

	c=2	c=4	c=6	c=8
a=2		2 9 4 7 5 3 6 1 8	2 7 6 9 5 1 4 3 8	
a=4	4 9 2 3 5 7 8 1 6			4 3 8 9 5 1 2 7 6
a=6	6 7 2 1 5 9 8 3 4			6 1 8 7 5 3 2 9 4
a=8		8 3 4 1 5 9 6 7 2	8 1 6 3 5 7 4 9 2	

These solutions include the four that were expected:
#1: The solution shown at the beginning of this paper (a=4,c=8).
#2: Solution #1 with rows 1 and 3 interchanged (a=2,c=6).
#3: Solution #1 with columns 1 and 3 interchanged (a=8,c=4).
#4: Solution #1 with rows and columns interchanged (a=6,c=2).

Four other solutions exist. Each can be created from one of the four expected solutions, rotating the array along a diagonal so that rows become columns and columns become rows:
#5: The two solutions for a=2 are rotations of each other.
#6: The two solutions for a=4 are rotations of each other.
#7: The two solutions for a=6 are rotations of each other.
#8: The two solutions for a=8 are rotations of each other.

We used algebra to find all the solutions to the puzzle and explain the relationships among them. Arithmetic couldn't do that.

Algebra:	Trial-and-error arithmetic:
1. proved how many solutions there are	1. proved there are several solutions but not how many
2. showed all solutions	2. showed some solutions
3. proved there are no more solutions	3. left us wondering if there were more solutions
4. avoided effort to find more solutions	4. required more effort if we wanted to find more solutions, did not tell us when further efforts might bear fruit, and did not tell us when further efforts would be fruitless because no more solutions existed
5. explained relationships among all solutions	5. showed relationships among the solutions discovered