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Square Root of 2: Irrational, Yes! Impractical, No!

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For the mathematically inclined person, irrational numbers such as √2 are fascinating, both from a historical perspective, and as a classic example of using the *reductio ad absurdum* proof, to prove the irrationality of √2. However, the majority of students, when introduced to the irrational number √2, might be excused for not being fascinated by it, and might believe that √2 has not much practical use. For instance, how many would believe that something we use everyday, like standard sizes of paper (the **"A" series**), have a very close relationship to √2?

I am sure, too, that most of us will remember using trial and error to make reduced size copies of an original, only to find the reduction missing some text, or leaving a lot of white space in one dimension or the other! In this article, I discuss the very important relationship between √2 and the **A** series of paper sizes, and how such a relationship helps us solve the problem of fitting two pages of an original onto a single-page copy.

But before discussing the **A** series of paper sizes, let's take a look at the following problem:
Suppose we want to reduce two 8.5" by 11" size pages, so as to get one 8.5" by 11" page copy. How could we do this?

If two such papers are laid next to each other with the longer sides touching/adjacent, we get a page with the dimensions 11" x 17". If we shrink by half the 11" x 17" paper, the dimensions of the reduced copy would be 5.5" x 8.5". Hence, we get a copy that does not preserve the original shape, because the width-to-length ratio of the original and that of the copy are not the same -- the larger (original) paper will have a width to length ratio of 11/17 ≈ 0.65, whereas the ratio is 0.77 for the 8.5" x 11" paper. How, then, could we get our desired, non-distorted, but reduced copy?

To get a better idea of what is involved in reducing two pages to a one-page copy without distortion, let us take a look at how the **A** series of paper sizes seem to have an ideal solution.

The A size/format Paper

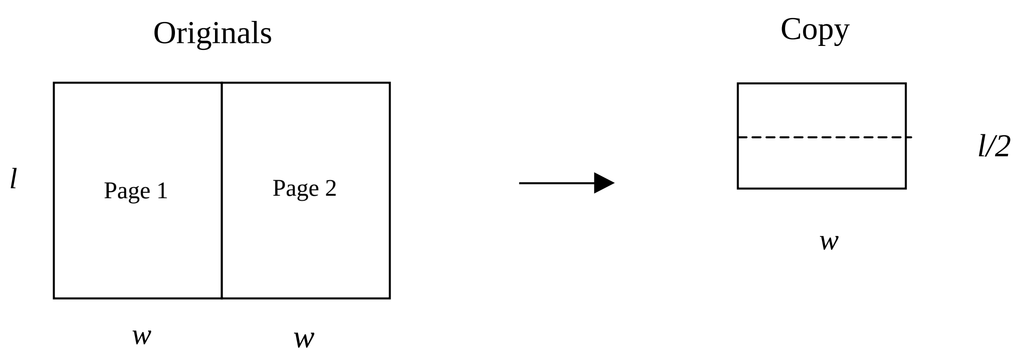
What is the **A4** size paper? **A4** is one of a series of international paper sizes based on metric measures, and the number √2. They range from **A0**, **A1**, **A2**, through **A10**, with **A0** being the largest, having an area of 1 m^2, **A1** being 0.5 m^2, **A2** being 0.25 m^2, and so on. The approximate dimensions (in mm) of the **A3**, **A4**, and **A5** size papers are, respectively, 297 x 420, 210 x 297, and 148 x 210.

The Problem

Let's say you are copying a long article, and you decide to save a lot of paper by copying two pages of the original onto one. (You also realize it will be easier to carry and store a copy that is half the weight and volume of the original.) So the problem boils down to copying two equally sized pages, having width w and length l, onto a single page, reducing the area of each original by half. To fit each original page onto the copy without distortion, you also want the copy to preserve the ***shape*** (that is, the length-to-width ratio) of the original pages. (A similar issue comes up when a movie image is re-formatted to fit on a television screen.) So, how would you reduce two pages to fit into one page, without distortion?

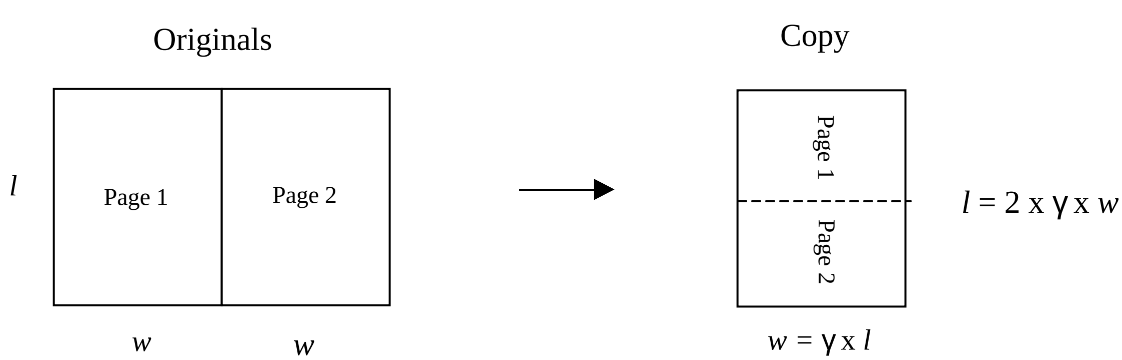
The Solution

At least two options seem available. One, if the two original pages are twice the width of the copy, we could shrink everything by half. However, doing this would shrink everything by half, including the length, and the copy would be distorted, as the shape is not preserved. The situation is illustrated in Figure 1.



<u>Figure 1</u>

An alternative possibility is to rotate by 90°, and then make the (doubled) width of the pages shrink down to the length of the desired page. Doing this also shrinks the length, so that it fits to the width. However, this only works if the ratio of width to length is appropriate. Let's call this width to length ratio γ. This situation is illustrated in Figure 2.



<u>Figure 2</u>

Each original page has width w and length l. For the copy, the length and width of each original has been reduced by a factor γ (which is <1). To preserve the shape (the length/width ratio) of each original page, the same reduction factor γ must be applied to both the width and length of the originals. [Unless $l > 2 \times w$, you are better off rotating the copies 90˚, since the longer dimension of each copied page will be w, and the shorter one, $l/2$.] Two equations describe the relationships between dimensions of the original and copy:

$$w = \gamma \; l \qquad (1)$$

$$l = 2 \; \gamma \; w \qquad (2)$$

Using (1) to eliminate w in (2), we get:

$$\gamma = 1/\sqrt{2} \;\; = \;\; \sqrt{2}/2 \qquad (3)$$

Substituting the result for γ back into equation (1), this says that to copy two originals onto a single page of the same size paper, while preserving the shapes of the originals, ***the ratio of the width to the length of the paper involved must be 1: √2!***

Discussion

Notice that the ratio of the width to length of **A3** size paper is 297/420 ≈ 0.71 ≈ 1/√2. If we compare the width of the **A4** size paper to its length, we get 210/297 ≈ 0.71 ≈ 1/√2 too. Similarly, for the **A5** size paper, its width and length are in the ratio 297/420 ≈ 0.71 ≈ 1/√2; this relationship holds true for all other **A** size/format papers as well.

But if you try the two-to-one copying trick with two 8.5” by 11” originals, it will not work. Dividing 8.5 by 11 gives about 0.77. Try the experiment yourself – you will either leave white space, or lose part of the original material.

A Related Property of the A size/format

There are further consequences of the **A** series’ special shape. The width of an **A3** size paper equals the length of an **A4** size paper, and the width of an **A4** size paper is the length of an **A5** size paper, and so on. This means that putting two **A4** size papers next to each other (with the lengths juxtaposed) will result in an **A3** size paper (see Figure 3), two **A3** size papers together will result in an **A2** size paper, and so on. For example, if two **A4** size papers (where each is 210 x 297) are put together, we will get an **A3** size paper (with dimensions 297 x 420). So, if the originals in Figure 2 were each **A3**s, you could cut the copy in half along the dashed line are get two **A4**s.

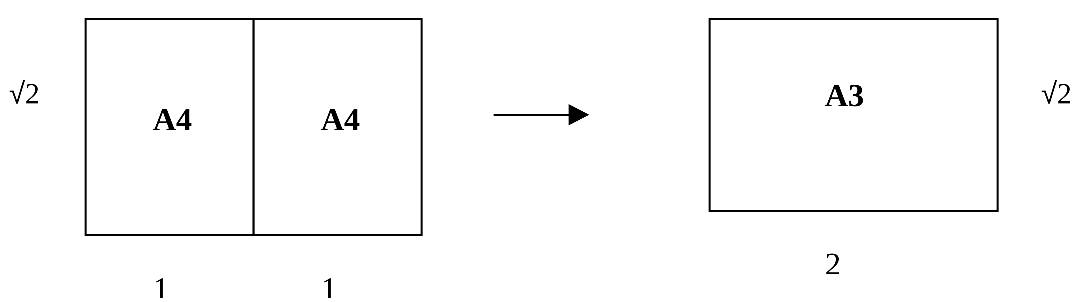


Figure 3

Note that, as expected, the width/length of **A4** paper is $1/\sqrt{2}$, and in the same units, for **A3** it is $\sqrt{2}/2 = 1/\sqrt{2}$. This shape is preserved when the area is doubled, just as it is when the area is halved.

The same does not apply to the common 8.5" by 11" size. If two such papers are laid next to each other with the longer sides touching/adjacent, we get a page with the familiar dimensions 11" x 17". (Having an 11" side for both sizes is handy for copy machine designers.) But the width to length ratio of this larger paper will be $11/17 \approx 0.65$, not the same as the 0.77 ratio for 8.5" x 11", as we stated earlier.

Extension

Digital cameras commonly use a 3:4 ratio (e. g. 1200 x 1600 pixels), possible to match the ratios used for most computer monitors. However, when making prints of the digital photos, the choices offered to consumers are usually the standard photo print sizes, namely 3 x 5, 4 x 6, 5 x 7, 8 x 10, etc. Now, the ratios 3:5, 4:6, 5:7, etc., of the standard photo print sizes, are all different from the ratio 3:4 of most digital photos. What, then would happen when getting standard prints from a digital photo? Would the enlargement/reduction preserve the shape of the original digital image? Are we getting the "actual" digital photo image, or does the "cropping" affect the final print? What happens, too, when enlargements/reductions are made between the standard photo print sizes themselves?

Further Information

For those interested in finding out more about not only the use of **A** series of paper sizes, but also those of the **B** series (for books, newspapers, and playing cards), and **C** series (for envelopes) paper sizes, please refer to the following URL:

http://www.cl.cam.ac.uk/~mgk25/iso-paper.html.